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Minimal length, maximal energy and black-hole remnants

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Abstract – In this paper, we investigate the impact of the generalized uncertainty principle (GUP), proposed by some approaches to quantum gravity such as string theory and doubly special relativity theories (DSR), on black-hole physics. It turns out that such a modification will give corrections to both the temperature and the entropy of black holes. In particular, the proposed GUP also changes the picture of Hawking radiation greatly when the size of black holes approaches the Planck scale. It prevents black holes from total evaporation, predicting the existence of black-holes remnants which may be considered as a candidate for dark matter.

Introduction. – The existence of a minimal length is one of the most interesting predictions of some approaches related to quantum gravity such as string theory as well as black-hole physics. This is a consequence of string theory since strings cannot interact at distances smaller than their size which yields *generalized uncertainty principle* (GUP) [1]. Black-hole (BH) physics suggests that the uncertainty relation should be modified near the Planck energy scale because of measuring the photons emitted from the black hole suffers from two major errors. The first one is the error by Heisenberg classical analysis and the second one is because the black-hole mass varies during the emission process and the radius of the horizon changes accordingly [1–5]. An interesting measure Gedanken experiment was proposed in [6] involving micro-black holes at the Planck scale of spacetime which leads to the GUP. This independent model depends on Heisenberg principle and Schwarzschild radius.

Recently, a new model of GUP was proposed in [7–9] which predicts a maximum observable momentum besides the existence of minimal measurable length and in consistency with doubly special relativity (DSR) theories, string theory and black-holes physics *and* ensures $[x_i, x_j] = 0 = [p_i, p_j]$ (via the Jacobi identity):

$$[x_i, p_j] = i\hbar\left[\delta_{ij} - \alpha\left(p\delta_{ij} + \frac{p_i p_j}{p}\right) + \alpha^2\left(p^2\delta_{ij} + 3p_i p_j\right)\right], \tag{1}$$

where $\alpha = \alpha_0/M_p c = \alpha_0 \ell_p/\hbar$, $M_p =$ Planck mass, $\ell_p =$ Planck length, and $M_p c^2 =$ Planck energy.

It was found that eq. (1) is approximately covariant under DSR transformations [10]. Since DSR transformations preserve both speed of light, and invariant energy scale [11], it is not surprising that eq. (1) implies the existence of minimum measurable length and maximum measurable momentum for a single particle. However, the existence of maximal energy bound does not make sense for macroscopic objects which have rest energies higher than the Planck mass. This paradox is usually referred to in the DSR literature as the soccer ball problem:

$$\Delta x \geqslant (\Delta x)_{min} \approx \alpha_0 \ell_p, \tag{2}$$

$$\Delta p \leqslant (\Delta p)_{max} \approx \frac{M_p c}{\alpha_0}. \tag{3}$$

The proposed GUP suggests that the space is discrete, and that all measurable lengths are quantized in units of a fundamental minimum measurable length (which can be the Planck length). A similar quantization of length was shown within the context of loop quantum gravity in [12] and discreteness of area was obtained in noncommutative

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geometry [13]. Recently, it was suggested in [14] that the GUP implications can be measured directly in quantum optics lab which confirm the theoretical predictions in [15,16].

Since the GUP modifies the fundamental commutator bracket between position and momentum, naturally it modifies the Hamiltonian and hence it affects a host of quantum phenomena, and it is important to make a quantitative study of these effects. In a series of earlier papers, the effects of GUP were investigated on atomic, condensed matter, preheating phase of the universe systems, black holes at LHC [8,15–19], the weak equivalence principle (WEP), the Liouville theorem (LT) in statistical mechanics [20]. It was found that the GUP can potentially explain the small observed violations of the WEP in neutron interferometry experiments [21] and also predicts a modified invariant phase space which is relevant to the Liouville theorem.

In this paper, we present a study for the impact of the GUP, proposed by some approaches to quantum gravity such as string theory and doubly special relativity theories (DSR) and which respect the Jacobi identity on black-hole physics, (see eqs. (1)). We calculate the corrections to both the temperature and the entropy of black holes.

An outline of this paper is as follows. In the second section, we review briefly how standard Hawking temperature can be obtained from the standard uncertainty principle In the third section, we investigate BH thermodynamics if the previously proposed GUP [7,9] is taken into consideration. We give our conclusions in the fourth section.

Hawking temperature —uncertainty relation connection. – In this section, we review the connection between standard Hawking temperature and uncertainty relation that has been proposed by Scardigli in [22] and was studied with GUP of [1] by Adler *et al.* in [23]. A BH could be imagined as a 3-dimensional sphere of size equal to twice the Schwarzschild radius, r_s. Since the Hawking radiation is a quantum process, so the emitted particle should obey the Heisenberg uncertainty relation. This leads to momentum-position uncertainty,

$$\Delta p \Delta x \geqslant \frac{\hbar}{2}, \tag{4}$$

where the uncertainty in position of emitted Hawking particle has its minimum value given by

$$\Delta x \approx 2 r_s = \frac{4GM}{c^2}, \tag{5}$$

Using eqs. (4), (5) the energy uncertainty of the emitted Hawking particle is given by

$$\Delta E \approx c\Delta p = c\frac{\hbar}{2\Delta x} \approx c\frac{\hbar}{4 r_s} = \frac{M_p c^2}{8}\left(\frac{M}{M_p}\right)^{-1}. \tag{6}$$

From now on, we can assume $m = \frac{M}{M_p}$, where m is the mass in units of the Planck mass and the Planck mass M_p is given by $M_p = \sqrt{\frac{\hbar c}{G}}$. As suggested by Scardigli in [20] and Adler *et al.* in [23], one can define the energy uncertainty ΔE as the energy of the emitted photon from the black hole. This argument implies that one can get the characteristic temperature of the emitted Hawking particle by just multiplying ΔE with a factor $\frac{1}{\pi}$ to give exactly the Hawking temperature [24]

$$T_H = \frac{1}{8\pi}\, M_p c^2\, m^{-1}. \tag{7}$$

The black-hole entropy can be calculated through the first law of black-hole thermodynamics:

$$\mathrm{d}M = \frac{1}{c^2} T \mathrm{d}S. \tag{8}$$

Using the mass in units of the Planck mass, m, one can rewrite eq. (8) as

$$\mathrm{d}S = M_p c^2 \frac{1}{T} \mathrm{d}m. \tag{9}$$

By integrating eq. (9) using eq. (7), one can obtain the the Bekenstein entropy [25] as follows:

$$S = 4\pi m^2. \tag{10}$$

The specific heat can be calculated using the thermodynamical relation

$$\mathcal{C} = T\frac{\partial S}{\partial T} = T\frac{\partial S}{\partial m}\frac{\partial m}{\partial T} = M_p c^2 \frac{\partial m}{\partial T}, \tag{11}$$

By differentiating eq. (7) and substituting this into eq. (11), the specific heat could be given by

$$\mathcal{C} = -8\pi m^2. \tag{12}$$

The Hawking temperature T_H can be used in the calculation of the emission rate. The emission rate might be calculated using the Stefan-Boltzmann law considering that the energy loss was dominated by photons. The emission rate of the black hole is

$$\frac{\mathrm{d}M}{\mathrm{d}t} \propto T^4, \tag{13}$$

The emission rate will be

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -\frac{\mu'}{t_p}\, m^{-2}, \tag{14}$$

where $t_p = \left(\frac{\hbar G}{c^5}\right)^{\frac{1}{(2)}}$ is the Planck time, and the form of μ can be found in [26]. The exact calculation should consider the squeezing of the fundamental cell in momentum space, which modifies the emission rate equation (14). However, one can neglect this effect in the first-order approximation [4,27].

The decay time of the black hole can be obtained by integrating eq. (14) to give

$$\tau = \left(\frac{1}{3}\right) \mu'^{-1} m_i^3\, t_p. \tag{15}$$

One can notice that the calculated Hawking temperature T_H, Bekenstein entropy S, specific heat $\mathcal{C}$, emission rate $\frac{\mathrm{d}m}{\mathrm{d}t}$, and decay time τ lead to *catastrophic evaporation* as $m \to 0$. This can be explained as follows. Since $\mathcal{C}=0$ only when $m=0$, the black hole will continue to radiate until $m=0$. But as the black hole approaches zero mass, its temperature approaches infinity with infinite radiation rate. This was just a brief summary for the Hawking radiation-uncertainty principle connection, and the catastrophic implications of the Hawking radiation as the black-hole mass approaches zero. In the next section, we study BH thermodynamics if GUP is taken into consideration. The end-point of the Hawking radiation is not catastrophic because GUP implies the existence of BH remnants at which the specific heat vanishes and, therefore, the BH cannot exchange heat with the surrounding space. The GUP prevents BHs from evaporating completely, just like the standard uncertainty principle prevents the hydrogen atom from collapsing [23,26].

GUP and BH thermodynamics. – In this section, we make an analysis of BH thermodynamics if GUP proposed in [7–9] is taken into consideration.

The emitted particles as Hawking radiation are mostly photons and standard model (SM) particles. For simplicity, we might assume from the kinetic theory of gases a cloud of points in velocity space, equally spread in all directions (there is no reason that a particle would prefer to be moving in the x-direction, say, rather than in the y-direction) and consider

$$p_1 \approx p_2 \approx p_3. \tag{16}$$

This assumption leads to

$$\begin{aligned} p^2 &= \sum_{i=1}^{3} p_i p_i \approx 3p_i^2, \\ \langle p_i^2 \rangle &\approx \frac{1}{3}\langle p^2 \rangle. \end{aligned} \tag{17}$$

Now, we want to find the relation between $\langle p^2 \rangle$ and Δp^2. We assume that we have a photon gas emitted from the BH like emission from a black body. Therefore, we may use Wien's law which gives a temperature corresponding to a peak emission at an energy given by

$$c\langle p \rangle = 2.821 \;\; T_H. \tag{18}$$

From the Hawking-uncertainty connection proposed by Adler *et al.* in [23] we have

$$T_H = \frac{1}{\pi}\; c\;\Delta p = \frac{1}{2.821}\; c \;\; \langle p \rangle. \tag{19}$$

We get the following relations using the relation $\langle p^2 \rangle = \Delta p^2 + \langle p \rangle^2$:

$$\begin{aligned} \langle p \rangle &= 2.821 \frac{1}{\pi}\; \Delta p = \sqrt{\mu}\; \Delta p, \\ \langle p^2 \rangle &= (1+\mu)\Delta p^2, \quad \text{where} \quad \mu = \left(2.821\; \frac{1}{\pi}\right)^2. \end{aligned} \tag{20}$$

We would like to find the corresponding inequality for eq. (1). Equation (1) gives by using the argument used in [26]

$$\Delta x \Delta p \geqslant \frac{\hbar}{2}\left[1 - \alpha\langle p \rangle - \alpha\left\langle \frac{p_i^2}{p} \right\rangle + \alpha^2\langle p^2 \rangle + 3\alpha^2\langle p_i^2 \rangle\right]. \tag{21}$$

Using the arguments of eqs. (17), (20), in the inequality (21), we get

$$\Delta x \Delta p \geqslant \frac{\hbar}{2}\left[1 - \alpha_0 \ell_p\left(\frac{4}{3}\right)\sqrt{\mu}\frac{\Delta p}{\hbar} + 2(1+\mu)\alpha_0^2 \ell_p^2 \frac{\Delta p^2}{\hbar^2}\right]. \tag{22}$$

The last inequality (and as far as we know the only one) follows from eq. (1).

By solving the inequality (22) as quadratic equation in Δp, we obtain

$$\begin{aligned} \frac{\Delta p}{\hbar} \geqslant{}& \frac{2\Delta x + \alpha_0 \ell_p\left(\frac{4}{3}\sqrt{\mu}\right)}{4(1+\mu)\alpha_0^2 \ell_p^2} \\ &\times \left[1 - \sqrt{1 - \frac{8(1+\mu)\alpha_0^2 \ell_p^2}{\left(2\Delta x + \alpha_0 \ell_p\left(\frac{4}{3}\right)\sqrt{\mu}\right)^2}}\right], \end{aligned} \tag{23}$$

where we considered only the negative sign $(-)$ solution which gives the standard uncertainty relation as $\frac{\ell_p}{\Delta x} \to 0$.

The modified Hawking temperature will be given by

$$\begin{aligned} T'_H ={}& \frac{1}{\pi\alpha_0^2}\frac{M_p c^2}{(1+\mu)}\left(2m + \frac{\alpha_0\sqrt{\mu}}{3}\right) \\ &\times \left[1 - \sqrt{1 - \frac{(1+\mu)\alpha_0^2}{2\left(2m + \frac{\alpha_0\sqrt{\mu}}{3}\right)^2}}\right] \end{aligned} \tag{24}$$

$$\begin{aligned} ={}& 2T_H\left(1 + \frac{\alpha_0\sqrt{\mu}}{6m}\right)^{-1} \\ &\times \left[1 + \sqrt{1 - \frac{(1+\mu)\alpha_0^2}{2\left(2m + \frac{\alpha_0\sqrt{\mu}}{3}\right)^2}}\right]^{-1}. \end{aligned} \tag{25}$$

The modified Hawking temperature is physical as far as the black-hole mass satisfies the following inequality:

$$(1+\mu)\;\alpha_0^2 \leqslant 2\left(2m + \frac{\alpha_0\;\sqrt{\mu}}{3}\right)^2. \tag{26}$$

This tells us the black hole should have minimum mass M_{min} given by

$$M_{min} = M_p\left(\sqrt{\frac{(1+\mu)}{2}} - \sqrt{\frac{\mu}{9}}\right)\frac{\alpha_0}{2}. \tag{27}$$

The endpoint of Hawking evaporation in the GUP case is characterized by a Planck-size remnant with maximum temperature

$$T_{max} \approx 2\left[\frac{\frac{3(1+\mu)}{2} + \sqrt{\frac{\mu(\mu+1)}{2}}}{\frac{3}{2} + \frac{7}{6}\mu}\right] T_H. \tag{28}$$

The emission rate can be calculated using the Stefan-Boltzmann law, using eqs. (13), (14), we get

$$\begin{aligned}\frac{\mathrm{d}m}{\mathrm{d}t} &= -16\frac{\mu'}{t_p}m^{-2}\left(1+\frac{\alpha_0\sqrt{\mu}}{6m}\right)^{-4}\\ &\times\left[1+\sqrt{1-\frac{(1+\mu)\alpha_0^2}{2\left(2m+\frac{\alpha_0\sqrt{\mu}}{3}\right)^2}}\right]^{-4}. \qquad (29)\end{aligned}$$

The entropy can be calculated from the first law of BH thermodynamics,

$$\begin{aligned}\mathrm{d}S &= 4\pi m\left(1+\frac{\alpha_0\sqrt{\mu}}{6m}\right)\\ &\times\left[1+\sqrt{1-\frac{(1+\mu)\alpha_0^2}{2\left(2m+\frac{\alpha_0\sqrt{\mu}}{3}\right)^2}}\right]\mathrm{d}m, \qquad (30)\end{aligned}$$

$$\begin{aligned}S &= \frac{2}{3}\pi\sqrt{\frac{-3\alpha_0^2\mu-3\alpha_0^2+2\alpha_0\sqrt{\mu}+12m}{\alpha_0\sqrt{\mu}+6m}}\\ &\times\left(-\frac{\alpha_0^2\sqrt{\mu}\left(3\alpha_0\mu+3\alpha_0-4\sqrt{\mu}\right)}{48\sqrt{2}}\right.\\ &\left.+\frac{3m^2}{\sqrt{2}}-\frac{\alpha_0 m\left(3\alpha_0\mu+3\alpha_0-8\sqrt{\mu}\right)}{8\sqrt{2}}\right)\\ &+2\pi m^2-\frac{1}{32}\pi\alpha_0^4(\mu+1)^2\log\left(-3\alpha_0^2\mu-3\alpha_0^2+4\alpha_0\sqrt{\mu}\right.\\ &+2\sqrt{2}\alpha_0\sqrt{\mu}\sqrt{\frac{-3\alpha_0^2\mu-3\alpha_0^2+2\alpha_0\sqrt{\mu}+12m}{\alpha_0\sqrt{\mu}+6m}}\\ &\left.+12\sqrt{2}m\sqrt{\frac{-3\alpha_0^2\mu-3\alpha_0^2+2\alpha_0\sqrt{\mu}+12m}{\alpha_0\sqrt{\mu}+6m}}+24m\right)\\ &+\frac{2}{3}\pi\alpha_0\sqrt{\mu}m, \qquad (31)\end{aligned}$$

$$\begin{aligned}S &= 4\pi m^2+\frac{4}{3}\pi\alpha_0\sqrt{\mu}m\\ &+\frac{1}{18}\alpha_0^2(\pi\mu-9\pi\mu m-9\pi m)+O\left(\alpha_0^3\right). \qquad (32)\end{aligned}$$

The specific heat has been calculated in the GUP case to give

$$\begin{aligned}\mathcal{C} &= -\pi\left(2m+\frac{\alpha_0\sqrt{\mu}}{3}\right)^2\sqrt{1-\frac{(1+\mu)\alpha_0^2}{2\left(2m+\frac{\alpha_0\sqrt{\mu}}{3}\right)^2}}\\ &\times\left[1+\sqrt{1-\frac{(1+\mu)\alpha_0^2}{2\left(2m+\frac{\alpha_0\sqrt{\mu}}{3}\right)^2}}\right]. \qquad (33)\end{aligned}$$

We note that the BH specific heat vanishes at the minimum BH mass. Therefore, the BH cannot exchange heat with the surrounding space and hence predicts the existence of black-hole remnants which may be considered as candidates for dark matter.

Conclusions. – In this paper we investigated the effect of GUP on the black-hole thermodynamics. We found that the GUP leads to a new mass-temperature relation and defines a minimum mass and maximum temperature for the black holes predicting the existence of black-hole remnants which may be considered as candidates for dark matter. In the future, it would be appropriate to generalize the calculations in extra dimensions to investigate the possibilities to see black holes at LHC.

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